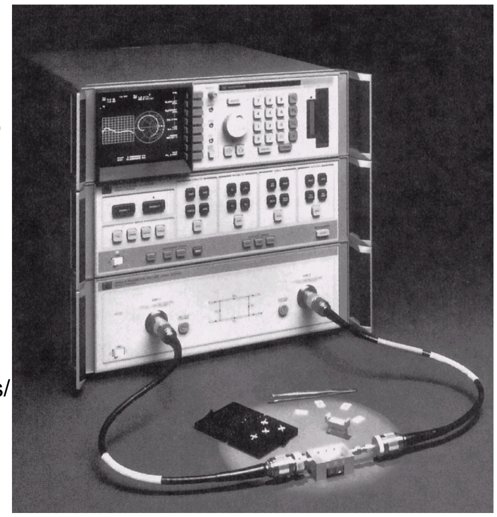
Scattering Parameters

Motivation

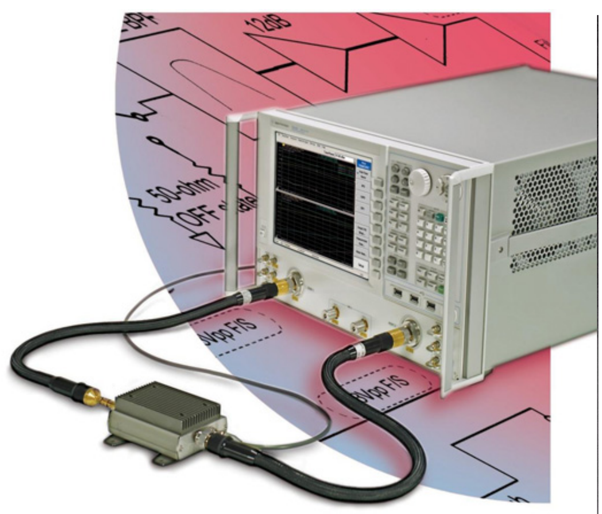
- Difficult to implement open and short circuit conditions in high frequencies measurements due to parasitic *L's* and *C's*
- Potential stability problems for active devices when measured in non-operating conditions
- Difficult to measure *V* and *I* at microwave frequencies
- Direct measurement of amplitudes/ power and phases of incident and reflected traveling waves



Scattering Parameters

Motivation

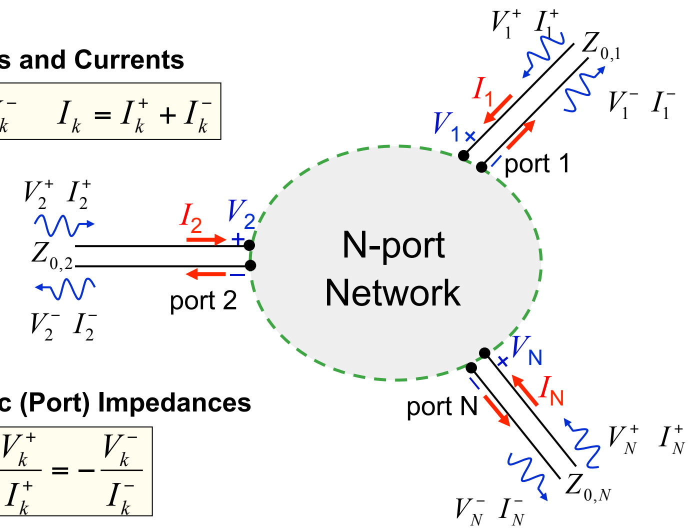
- Difficult to implement short circuit condition frequencies measure parasitic *L's* and *C's*
- Potential stability pro active devices when non-operating conditi
- Difficult to measure V microwave frequenci
- Direct measurement power and phases of reflected traveling wa



General Network Formulation

Port Voltages and Currents

$$V_k = V_k^+ + V_k^- \qquad I_k = I_k^+ + I_k^-$$

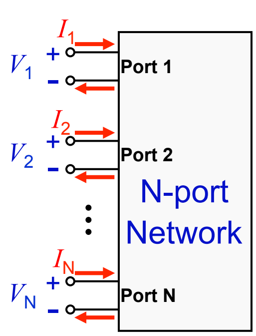


Characteristic (Port) Impedances

$$Z_{0,k} = \frac{V_k^+}{I_k^+} = -\frac{V_k^-}{I_k^-}$$

Note: all current components are defined positive with direction into the positive terminal at each port

Impedance Matrix

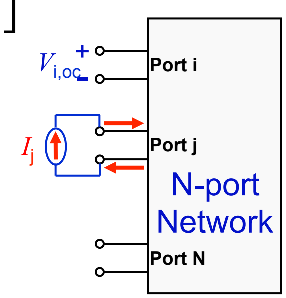


$$\begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \vdots & & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix}$$

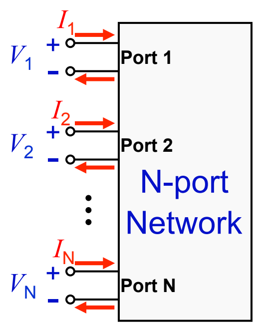
$$[V] = [Z][I]$$

Open-Circuit Impedance Parameters

$$Z_{ij} = \left. \frac{V_{i,oc}}{I_j} \right|_{I_k = 0 \text{ for } k \neq j}$$



Admittance Matrix

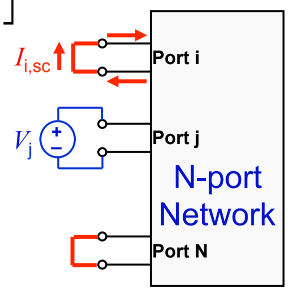


$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & & \vdots \\ Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$

$$[I] = [Y][V]$$

Short-Circuit Admittance Parameters

$$Y_{ij} = \left.\frac{I_{i,sc}}{V_j}\right|_{V_k=0 \text{ for } k \neq j}$$



The Scattering Matrix

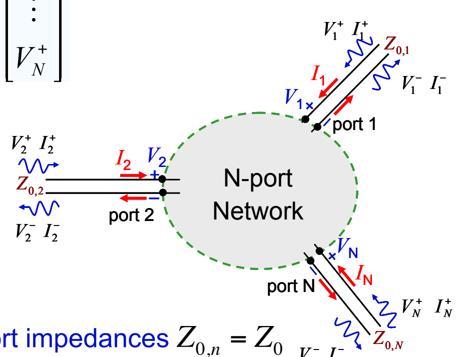
The scattering matrix relates incident and reflected voltage waves at the network ports as (assume $Z_{0,n} = Z_0$):

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \\ V_N^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \\ V_N^+ \end{bmatrix} \quad \text{or} \quad [V^-] = [S][V^+]$$

with voltage and current at port n:

$$V_n = V_n^+ + V_n^-$$

$$I_n = I_n^+ + I_n^- = \left(V_n^+ - V_n^-\right)/Z_0$$



Note: S-parameters depend on port impedances $Z_{0,n} = Z_0$

The Scattering Matrix

The scattering matrix relates incident and reflected voltage waves at the network ports as (assume $Z_{0,n} = Z_0$):

$$\begin{bmatrix} V_1^- \\ V_2^- \\ \vdots \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & & & \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \\ \vdots \end{bmatrix}$$

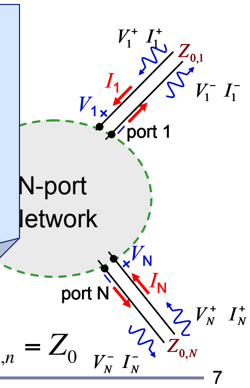
or $[V^-] = [S][V^+]$

with v...

$= (V_n^+ - V_n^-)/Z_0$

Interpretation?

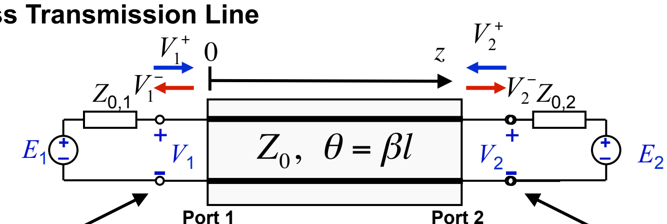
Power relationships?



Note: S-parameters depend on port impedances $Z_{0,n} = Z_0$

Transmission Line Basics

Lossless Transmission Line



$$V_1 = V_1^+ + V_1^-$$
$$I_1 = I_1^+ + I_1^-$$

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$
$$I(z) = I_0^+ e^{-j\beta z} + I_0^- e^{+j\beta z}$$

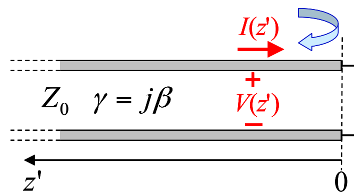
$$V_2 = V_2^+ + V_2^-$$
$$I_2 = I_2^+ + I_2^-$$

Phase Constant: $\beta = \omega\sqrt{LC}$

Characteristic Impedance: $Z_0 = \sqrt{\frac{L}{C}} = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$

Net Power Flow on Lossless Line

$Z_0 = R_0 (real)$



$\Gamma_L = \dfrac{V_0^-}{V_0^+}$

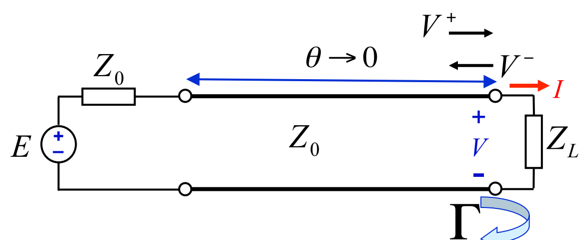
$$V(z') = V_0^+\left(e^{+j\beta z'} + \Gamma_L e^{-j\beta z'}\right)$$

$$I(z') = \frac{V_0^+}{Z_0}\left(e^{+j\beta z'} - \Gamma_L e^{-j\beta z'}\right)$$

$$P_{\text{ave}}(z) = \tfrac{1}{2}\text{Re}\left\{V(z)\left(I(z)\right)^*\right\} = \frac{\left|V_0^+\right|^2}{2R_0}\left[1 - \left|\Gamma_L\right|^2\right] = \text{const.}$$

Generalized Scattering Parameters

considerations and definitions

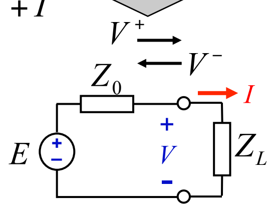


$V^+ = Z_0 I^+$ $\quad V^- = -Z_0 I^-$

$V = V^+ + V^-$ $\quad I = I^+ + I^-$

$$V^+ = \tfrac{1}{2}\left(V + Z_0 I\right)$$

$$V^- = \tfrac{1}{2}\left(V - Z_0 I\right)$$



$$V^+ = \frac{E}{2}$$

$$\Gamma = \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$V = \frac{Z_L}{Z_L + Z_0}E = \tfrac{1}{2}\left(1 + \Gamma\right)E$$

$$P^+ = \tfrac{1}{2}\text{Re}\left\{V^+(I^+)^*\right\} = \tfrac{1}{2}\left|V^+\right|^2 / R_0 = P_{\text{max}}$$

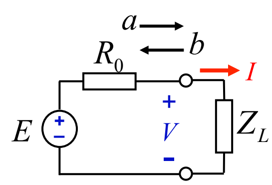
(assuming Z_0 is real $Z_0 = R_0$)

Normalized Wave Quantities

- It is useful to express power P without characteristic impedance (port impedance) $Z_0 = R_0$ (but P still depends on R_0)

$$P^+ = \tfrac{1}{2}\mathrm{Re}\left\{V^+ (I^+)^*\right\} = \tfrac{1}{2}\left|V^+\right|^2 \Big/ R_0 \quad \Rightarrow \quad a = \frac{V^+}{\sqrt{R_0}}$$

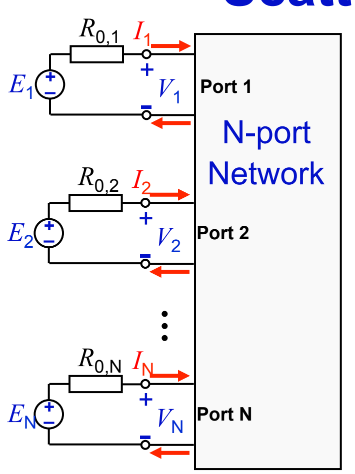
$$P^- = -\tfrac{1}{2}\mathrm{Re}\left\{V^- (I^-)^*\right\} = \tfrac{1}{2}\left|V^-\right|^2 \Big/ R_0 \quad \Rightarrow \quad b = \frac{V^-}{\sqrt{R_0}}$$



$$P_L = P^+ - P^- = \frac{\left|V^+\right|^2}{2R_0} - \frac{\left|V^-\right|^2}{2R_0}$$
$$= \tfrac{1}{2}\left\{|a|^2 - |b|^2\right\}$$

(assuming real Z_0)

Scattering Matrix



$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

$$S_{ij} = \left.\frac{b_i}{a_j}\right|_{a_k = 0 \text{ for } k \neq j}$$

$$S_{ij} = \left.\frac{V_i^- \big/ \sqrt{R_{0,i}}}{E_j \big/ \left(2\sqrt{R_{0,j}}\right)}\right|_{E_k = 0 \text{ for all } k \neq j} = \left.\frac{V_i \big/ \sqrt{R_{0,i}}}{E_j \big/ \left(2\sqrt{R_{0,j}}\right)}\right|_{E_k = 0 \text{ for all } k \neq j} \qquad (i \neq j)$$

Scattering Parameters

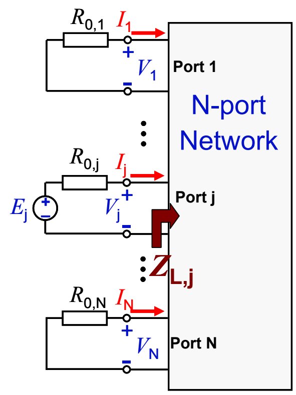
Physical meaning of $\left|S_{ij}\right|^2$ ($\mathbf{Z}_0 = \mathbf{R}_0$ = real)

$$\left|S_{ij}\right|^2 = \left.\frac{P_i^-}{P_{\max,j}}\right|_{\substack{E_k=0\\ k\neq j}} = \left.\frac{\text{actual power leaving port } i}{\text{maximum power from port } j}\right|_{\substack{E_k=0\\ k\neq j}} \quad (i \neq j)$$

Physical meaning of $\left|S_{jj}\right|^2$

$$S_{jj} = \left.\frac{b_j}{a_j}\right|_{\substack{a_k=0\\ k\neq j}} = \frac{V_j^- \big/ \sqrt{R_{0,j}}}{V_j^+ \big/ \sqrt{R_{0,j}}} = \frac{Z_{L,j} - R_{0,j}}{Z_{L,j} + R_{0,j}}$$

$$P_{L,j} = P_j^+ - P_j^- = P_{\max}\left\{1 - \left|S_{jj}\right|^2\right\}$$



Relation to Z-Matrix

Impedance matrix:

$$\mathbf{V} = \mathbf{Z}\,\mathbf{I} \qquad \text{with} \quad \mathbf{Z} = \begin{bmatrix} Z_{11} & \cdots & Z_{1N} \\ \vdots & & \vdots \\ Z_{N1} & \cdots & Z_{NN} \end{bmatrix}$$

Express **V**,**I** in terms of **a** and **b**

$$\mathbf{R}_0^{1/2}(\mathbf{a}+\mathbf{b}) = \mathbf{Z}\,\mathbf{R}_0^{-1/2}(\mathbf{a}-\mathbf{b}) \quad \mathbf{b} = (\mathbf{Z}_n + \mathbf{U})^{-1}(\mathbf{Z}_n - \mathbf{U})\mathbf{a}$$

with normalized impedance matrix $\mathbf{Z}_n = \mathbf{R}_0^{-1/2}\,\mathbf{Z}\,\mathbf{R}_0^{-1/2}$

($\mathbf{Z}_0 = \mathbf{R}_0$ = real)

and port impedance matrix $\mathbf{R}_0^{1/2} = \begin{bmatrix} \sqrt{R_{0,1}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{R_{0,N}} \end{bmatrix}$ $\quad \mathbf{R}_0^{-1/2} = \left(\mathbf{R}_0^{1/2}\right)^{-1}$

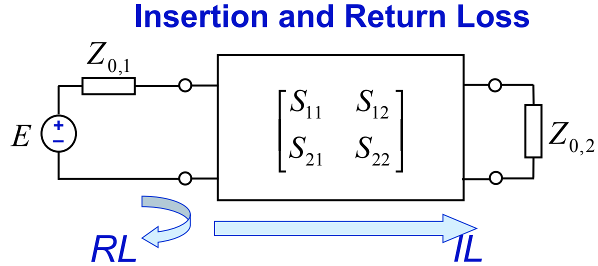
⇨ $$\mathbf{S} = (\mathbf{Z}_n + \mathbf{U})^{-1}(\mathbf{Z}_n - \mathbf{U}) = (\mathbf{Z}_n - \mathbf{U})(\mathbf{Z}_n + \mathbf{U})^{-1}$$

Scattering Parameters

- Port n is said to be matched when it is terminated with a load having the same impedance as the port impedance $Z_{0,n}$.
- Often, all port impedances are chosen to be equal and $Z_{0,n}$ = 50 Ω.
- The values of the scattering (S-) parameters depend on the chosen port impedances.
- S-parameters can be algebraically renormalized to different and unequal port impedances. (see later)

Two-Port Networks

Insertion and Return Loss



Return Loss

indicates the extend of mismatch in a network in dB

port 1: $RL = -20\log_{10}|S_{11}| \quad \text{in dB}$

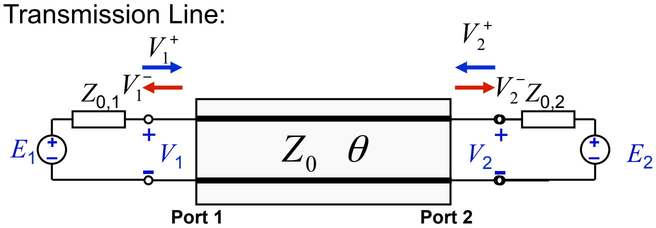
Insertion Loss

measure of transmitted fraction of power in dB

from port 1 to port 2: $IL = -20\log_{10}|S_{21}| \quad \text{in dB}$

Example

Lossless Transmission Line:



If $Z_{0,1} = Z_{0,2} = Z_0$, the scattering parameters can be easily obtained by inspection:

$$S_{11} = S_{22} = 0 \qquad S_{12} = S_{21} = e^{-j\theta}$$

$$[U] \pm [S] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pm \begin{bmatrix} 0 & e^{-j\theta} \\ e^{-j\theta} & 0 \end{bmatrix}$$

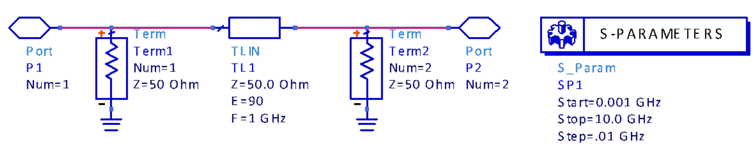
$$([U]-[S])^{-1} = \begin{bmatrix} 1 & -e^{-j\theta} \\ -e^{-j\theta} & 1 \end{bmatrix}^{-1} = \frac{1}{1-e^{-j2\theta}} \begin{bmatrix} 1 & e^{-j\theta} \\ e^{-j\theta} & 1 \end{bmatrix}$$

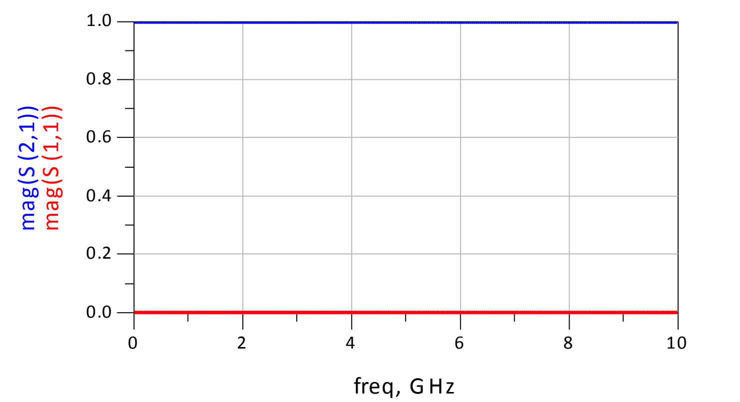
$$[Z] = Z_0([U]+[S])([U]-[S])^{-1} =$$

$$= \frac{Z_0}{1-e^{-j2\theta}} \begin{bmatrix} 1 & e^{-j\theta} \\ e^{-j\theta} & 1 \end{bmatrix} \begin{bmatrix} 1 & e^{-j\theta} \\ e^{-j\theta} & 1 \end{bmatrix} =$$

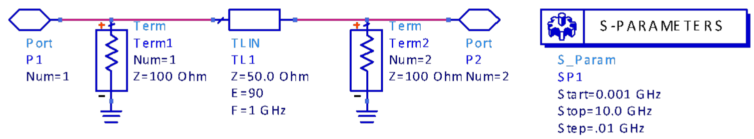
$$= \frac{Z_0}{1-e^{-j2\theta}} \begin{bmatrix} 1+e^{-j2\theta} & 2e^{-j\theta} \\ 2e^{-j\theta} & 1+e^{-j2\theta} \end{bmatrix} = \begin{bmatrix} -jZ_0 \cot\theta & -jZ_0/\sin\theta \\ -jZ_0/\sin\theta & -jZ_0 \cot\theta \end{bmatrix}$$

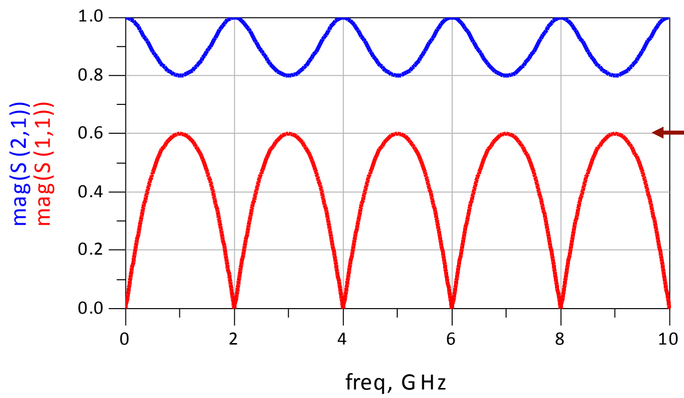
50Ω Transmission Line - 50Ω References





50Ω Transmission Line - 100Ω References





$$Z_{in} = \left(50^2/100\right)\Omega$$
$$= 25\Omega$$

$$S_{11} = \frac{25-100}{25+100} = -0.6$$

Properties of S-Parameters

Reciprocal networks:

$S_{ij} = S_{ji}$ or $[S] = [S]^T$ **Matrix symmetry!**

Symmetrical networks:

$S_{ii} = S_{jj}$ and $S_{ij} = S_{ji}$ **Electrical Symmetry and Matrix symmetry!**

Lossless networks:

For a lossless passive network the scattering matrix [S] is unitary:

transpose complex-conjugate

$$[S]^T [S]^* = [U]$$

Example: two-port network

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \Rightarrow \quad [S]^T [S]^* = ?$$

Properties of S-Parameters

Reciprocal networks:

$$S_{ij} = S_{ji} \quad \text{or} \quad [S] = [S]^T$$

Matrix symmetry!

Symmetrical networks:

$$S_{ii} = S_{jj} \quad \text{and} \quad S_{ij} = S_{ji}$$

Electrical Symmetry and Matrix symmetry!

Los…ss networks:

For a lossles…] is unitary:

PHYSICAL SYMMETRY?

Example: two-port networ…

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \Rightarrow [S]^T [S]^* = ?$$

Lossless Two-Port Networks

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad [S]^T = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} \qquad [S]^* = \begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix}$$

Then

$$[S]^T [S]^* = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} \begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} = \begin{bmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{11}S_{12}^* + S_{21}S_{22}^* \\ S_{12}S_{11}^* + S_{22}S_{21}^* & |S_{12}|^2 + |S_{22}|^2 \end{bmatrix}$$

From unitary condition follows:

$$|S_{11}|^2 + |S_{21}|^2 = 1 = |S_{12}|^2 + |S_{22}|^2$$

$$S_{12}S_{11}^* + S_{22}S_{21}^* = 0 = S_{11}S_{12}^* + S_{21}S_{22}^*$$

Example: lossless TL

$$[S] = \begin{bmatrix} 0 & e^{-j\theta} \\ e^{-j\theta} & 0 \end{bmatrix}$$

$$|S_{11}|^2 + |S_{21}|^2 = |0|^2 + |e^{-j\theta}|^2 = 1$$

$$S_{12}S_{11}^* + S_{22}S_{21}^* = e^{-j\theta} \cdot 0 + e^{-j\theta} \cdot 0 = 0$$

Lossless Two-Port Networks

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad [S]^T = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix} \qquad [S]^* = \begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix}$$

Then

$$[S]^T[S]^* = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix}\begin{bmatrix} S_{11}^* & S_{12}^* \\ S_{21}^* & S_{22}^* \end{bmatrix} = \begin{bmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{11}S_{12}^* + S_{21}S_{22}^* \\ S_{12}S_{11}^* + S_{22}S_{21}^* & |S_{12}|^2 + |S_{22}|^2 \end{bmatrix}$$

for passive (lossy) networks

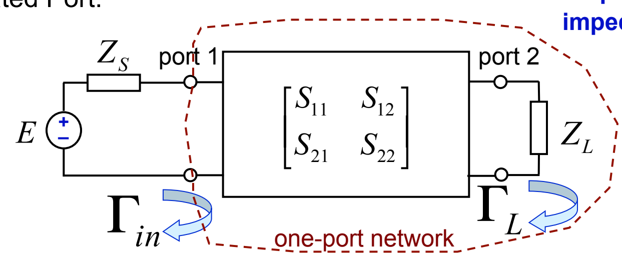
$$|S_{11}|^2 + |S_{21}|^2 \le 1 \ge |S_{12}|^2 + |S_{22}|^2$$

$$S_{12}S_{11}^* + S_{22}S_{21}^* = 0 = S_{11}S_{12}^* + S_{21}S_{22}^*$$

Applications

Terminated Port:

All port (reference) impedances are $Z_{0,ref}$



$$V_2^+ = \Gamma_L V_2^-$$

$$V_1^- = S_{11}V_1^+ + S_{12}V_2^+$$
$$V_2^- = S_{21}V_1^+ + S_{22}V_2^+$$

$$V_2^+ = \Gamma_L S_{21}V_1^+ + \Gamma_L S_{22}V_2^+ \qquad V_2^+ = \frac{\Gamma_L S_{21}}{1-\Gamma_L S_{22}}V_1^+$$

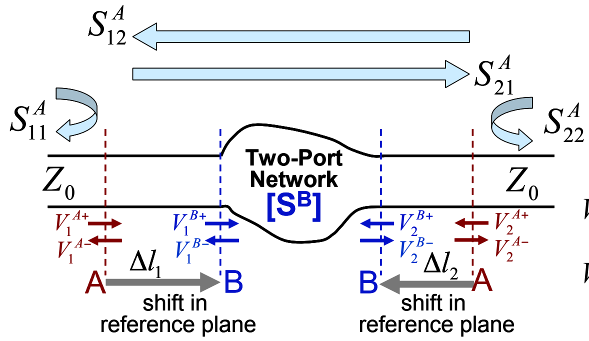
$$\Gamma_{in} = S_{11} + \frac{\Gamma_L S_{12}S_{21}}{1-\Gamma_L S_{22}}$$

$(Z_s = Z_{0,ref})$

special case: $Z_L=0$ ➔ $\Gamma_L = -1$

$$\Gamma_{in} = S_{11} - \frac{S_{12}S_{21}}{1+S_{22}}$$

Shift in Reference Plane



$$\theta_1 = \beta \Delta l_1 \qquad \theta_2 = \beta \Delta l_2$$

$$V_1^{B+} = e^{-j\theta_1} V_1^{A+} \qquad V_2^{B+} = e^{-j\theta_2} V_2^{A+}$$

$$V_1^{B-} = e^{j\theta_1} V_1^{A-} \qquad V_2^{B-} = e^{j\theta_2} V_2^{A-}$$

$$\begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix} \begin{bmatrix} V_1^{B-} \\ V_2^{B-} \end{bmatrix} = \begin{bmatrix} S_{11}^A & S_{12}^A \\ S_{21}^A & S_{22}^A \end{bmatrix} \begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix} \begin{bmatrix} V_1^{B+} \\ V_2^{B+} \end{bmatrix}$$

$$\begin{bmatrix} S_{11}^B & S_{12}^B \\ S_{21}^B & S_{22}^B \end{bmatrix} = \begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix} \begin{bmatrix} S_{11}^A & S_{12}^A \\ S_{21}^A & S_{22}^A \end{bmatrix} \begin{bmatrix} e^{j\theta_1} & 0 \\ 0 & e^{j\theta_2} \end{bmatrix}$$

S-Matrix Renormalization

$$\mathbf{S} = (\mathbf{Z}_n + \mathbf{U})^{-1}(\mathbf{Z}_n - \mathbf{U}) = (\mathbf{Z}_n - \mathbf{U})(\mathbf{Z}_n + \mathbf{U})^{-1}$$

$$\mathbf{Z}_n = (\mathbf{U} + \mathbf{S})(\mathbf{U} - \mathbf{S})^{-1}$$

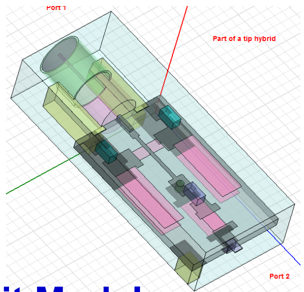
$$\mathbf{Z}_n^{\text{new}} = \mathbf{R}_{0,\text{new}}^{-1/2}\, \mathbf{Z}\, \mathbf{R}_{0,\text{new}}^{-1/2} = \mathbf{F}\, \mathbf{Z}_n^{\text{old}}\, \mathbf{F}$$

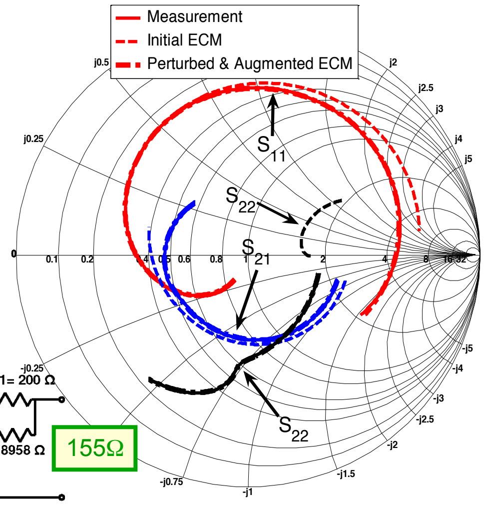
($\mathbf{Z}_0 = \mathbf{R}_0$ = real)

Renormalization matrix

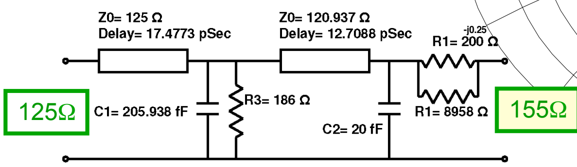
$$\mathbf{F} = \begin{bmatrix} \sqrt{R_{0,1}^{\text{old}} / R_{0,1}^{\text{new}}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{R_{0,N}^{\text{old}} / R_{0,N}^{\text{new}}} \end{bmatrix}$$

Example



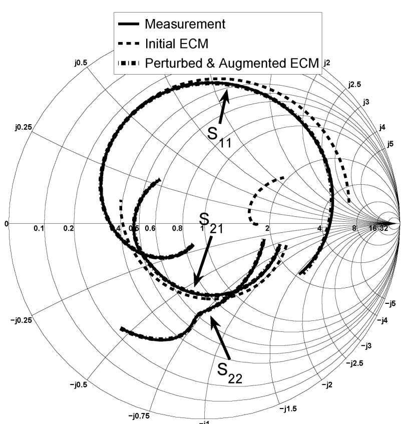


Circuit Model

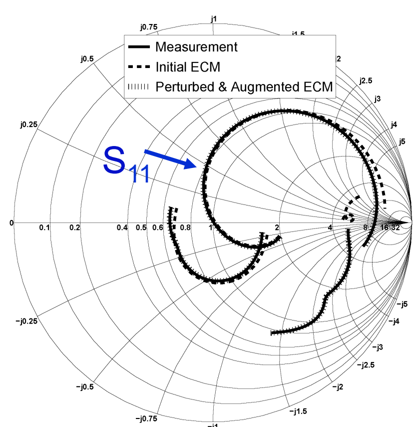


Comparison of Different Port Normalizations

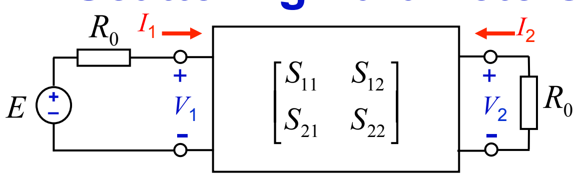
$Z_{0,1} = 125\Omega \quad Z_{0,2} = 155\Omega$



$Z_{0,1} = Z_{0,2} = 50\Omega$



Voltage Transfer Function from Scattering Parameters



$$V_1^+ = \tfrac{1}{2}\left(V_1 + Z_0 I_1\right) \qquad V_2^- = \tfrac{1}{2}\left(V_2 - Z_0 I_2\right)$$

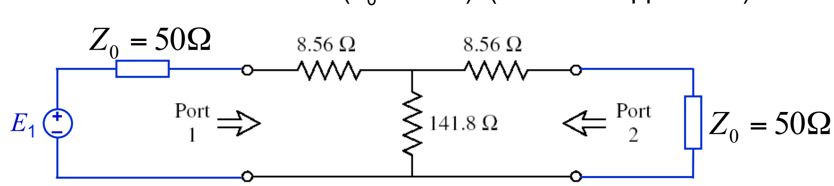
$$Z_{in} = R_0 \frac{1+S_{11}}{1-S_{11}} \qquad I_1 = \frac{V_1}{Z_{in}} = \frac{V_1}{R_0}\frac{1-S_{11}}{1+S_{11}}$$

$$S_{21} = \frac{V_2^-}{V_1^+} = \frac{V_2 - R_0 I_2}{V_1 + R_0 I_1} = \frac{2V_2}{V_1\left(1+\dfrac{1-S_{11}}{1+S_{11}}\right)} = \cdots = \frac{V_2}{V_1}\left(1+S_{11}\right)$$

$$\Rightarrow \quad \boxed{\frac{V_2}{V_1} = \frac{S_{21}}{1+S_{11}}}$$

Example

Matched 3dB attenuator (Z_0 = 50 Ω) (Ref: Pozar pp. 175/6)

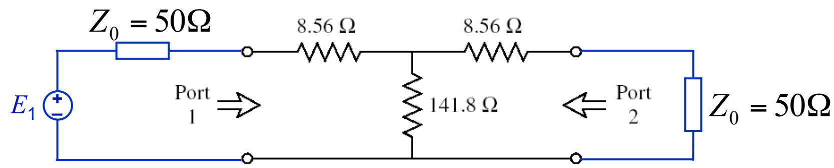


$$V_1 = E_1 \frac{Z_{in,1}}{Z_{in,1} + Z_0} = \tfrac{1}{2} E_1 \qquad Z_{in,1} = 41.444\Omega + 8.56\Omega \approx 50\Omega$$

$$V_2 = V_1 \left(\frac{41.444}{41.444+8.56}\right)\left(\frac{50}{50+8.56}\right) = 0.7077\, V_1$$

$$S_{11} = S_{22} = 0 \qquad S_{12} = S_{21} = 0.7077 \qquad IL = -20\log_{10}\left|S_{21}\right| = 3\text{dB}$$

Example using Z-Matrix



$$\mathbf{Z} = \begin{bmatrix} 150.36 & 141.80 \\ 141.80 & 150.36 \end{bmatrix} \Omega$$

$Z_{0,1} = Z_{0,2} = 50\Omega$

$$\mathbf{Z}_n = \mathbf{R}_0^{-1/2}\,\mathbf{Z}\,\mathbf{R}_0^{-1/2} = \begin{bmatrix} 3.0072 & 2.8360 \\ 2.8360 & 3.0072 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 0.7077 \\ 0.7077 & 0 \end{bmatrix}$$

$Z_{0,1} = 50\Omega \quad Z_{0,2} = 100\Omega$

$$\mathbf{Z}_n = \mathbf{R}_0^{-1/2}\,\mathbf{Z}\,\mathbf{R}_0^{-1/2} = \begin{bmatrix} 3.0072 & 2.0054 \\ 2.0054 & 1.5036 \end{bmatrix} \qquad S = \begin{bmatrix} 0.1670 & 0.6672 \\ 0.6672 & -0.3333 \end{bmatrix}$$

Attenuator terminated in Z_L=100Ω and S-Parameters wrt $Z_{0,1}$=$Z_{0,2}$=50Ω

$$\Gamma_L = \frac{Z_L - Z_{0,2}}{Z_L + Z_{0,2}} = \frac{100-50}{100+50} = \frac{1}{3}$$

$$\Gamma_{in} = S_{11} + \frac{\Gamma_L S_{12} S_{21}}{1 - \Gamma_L S_{22}} = \Gamma_L S_{12} S_{21} = \frac{1}{3}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 0.167$$

Attenuator terminated in Z_L=100Ω and S-Parameters wrt $Z_{0,1}$=50Ω $Z_{0,2}$=100Ω

$$\Gamma_L = \frac{Z_L - Z_{0,2}}{Z_L + Z_{0,2}} = 0$$

$$\Gamma_{in} = S_{11} + \frac{\Gamma_L S_{12} S_{21}}{1 - \Gamma_L S_{22}} = S_{11} = 0.167$$

50Ω

$$\mathbf{Z}_n = \mathbf{R}_0^{-1/2}\,\mathbf{Z}\,\mathbf{R}_0^{-1/2} = \begin{bmatrix} 3.0072 & 2.8360 \\ 2.8360 & 3.0072 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & 0.7077 \\ 0.7077 & 0 \end{bmatrix}$$

$Z_{0,1} = 50\Omega \quad Z_{0,2} = 100\Omega$

$$\mathbf{Z}_n = \mathbf{R}_0^{-1/2}\,\mathbf{Z}\,\mathbf{R}_0^{-1/2} = \begin{bmatrix} 3.0072 & 2.0054 \\ 2.0054 & 1.5036 \end{bmatrix} \qquad S = \begin{bmatrix} 0.1670 & 0.6672 \\ 0.6672 & -0.3333 \end{bmatrix}$$

Conversion between Network Parameters

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\lvert Y\rvert}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\lvert Y\rvert}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\lvert Y\rvert}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\lvert Y\rvert}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\lvert Z\rvert}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\lvert Z\rvert}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\lvert Z\rvert}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\lvert Z\rvert}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\lvert Z\rvert}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\lvert Y\rvert}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

Properties of Network Parameters

- **Symmetric Two-Port Network**

$$Z_{11} = Z_{22} \qquad Y_{11} = Y_{22} \qquad S_{11} = S_{22} \qquad A = D$$

assuming the same port impedances

- **Reciprocal Network**

$$Z_{ij} = Z_{ji} \qquad Y_{ij} = Y_{ji} \qquad S_{ij} = S_{ji} \qquad AD - BC = 1$$

- **Lossless Network**

$$\mathrm{Re}\{Z_{ij}\} = 0 \qquad \mathrm{Re}\{Y_{ij}\} = 0 \qquad \mathbf{S}^T\mathbf{S}^* = \mathbf{I} \qquad \begin{matrix}\mathrm{Re}\{B,C\} = 0 \\ \mathrm{Im}\{A,D\} = 0\end{matrix}$$

e.g. $|S_{11}|^2 + |S_{21}|^2 = 1$